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STABILITY OF THE VAPOR FILM DURING FILM BOILING FROM HORIZONTAL CYLINDERS

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Abstract—The stability of the vapor film formed on horizontal cylinders during stable film boiling is studied. The wavelength of the fastest growing perturbation is compared to the observed spacing of bubbles detaching from the vapor–liquid interface. A three-dimensional variational approach leads to results which are in good agreement with experiment for a fluid in a normal gravitational field. The effect of Coriolis forces on data obtained in a centrifugal force field is indicated.

INTRODUCTION

It has been observed that during stable film boiling vapor bubbles are released from the vapor–liquid interface covering a horizontal cylinder. The frequency of formation and the spacing of these bubbles are of major importance to determine the minimum heat flux for stable film boiling since the latter consists mainly of the heat of evaporation carried away by the detaching bubbles.

It was realized by Zuber & Tribus (1958) and Zuber (1959) that the Taylor instability of the film interface is responsible for the production of the bubbles. As summarized in the work of Lienhard & Sun (1970) several investigators studied the stability of the vapor–liquid interface on horizontal cylinders. The best results were obtained by Lienhard & Wong (1964). They studied the growth rate of a perturbation of the interface which is zero on the bottom and maximum on top of the interface while varying sinusoidally in the axial direction, as shown in figure 1a. This is in contrast with the symmetrical perturbation studied by other investigators (figure 1b). Lienhard & Wong (1964) started from a force balance at the top of the cylinder in which surface tension forces are given some average value over the circumference of the interface, thus eliminating the angular dependence. This two-dimensional analysis resulted in the determination of the wavelength of the perturbation with the maximum growth rate, or "most dangerous" wavelength λ_d, given by:

$$\lambda_d = \frac{3{\cdot}4641\pi}{\left[\frac{g(\rho_f-\rho_g)}{\sigma}+\frac{1}{2R^2}\right]^{1/2}} \qquad [1]$$

in which g is the gravitational constant, R the radius of the interface, ρ_f and ρ_g are the densities of the liquid and the gas respectively and σ is the surface tension between them. This theoretically determined λ_d is somewhat smaller than the experimentally observed one. Lienhard & Sun (1970) compared their results obtained in a normal gravitational field and in a centrifugal field with the above equation, claiming good agreement if a band of wavelengths around the "most dangerous" one is considered.

The purpose of the present work is to derive a more accurate expression for the "most dangerous" wavelength by means of a three-dimensional analysis. Recourse will be taken to a variational approach to interfacial stability which will be the subject of the next section. Subsequently this variational approach will be taken to study the stability of a horizontal cylindrical vapor film subjected to a normal gravitational field. This will be followed by a section on the effect of Coriolis forces acting in a centrifugal force field. Finally in the last section the conclusions will be drawn from the results obtained in this study.

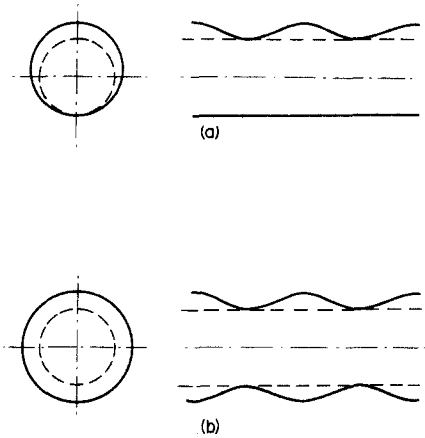

Figure 1. Asymmetrical perturbation (a) and symmetrical perturbation (b) of a cylindrical interface.

VARIATIONAL APPROACH TO STABILITY PROBLEMS

The analysis of Lienhard & Wong (1964) consists of perturbing the vapor–liquid interface and studying the development in time of the displacement by means of a balance of forces acting at the top of the cylinder. Instead of using this perturbation technique it is possible to determine the stability of a vapor–liquid interface by means of a variational analysis. A detailed treatment of this approach can be found in Berghmans (1972). In this section the variational technique will be outlined. Whether it is possible to apply it to the film stability problem will be analyzed in the following section.

Consider a system of two stagnant, immiscible, incompressible and inviscid fluids which may be subject to volume forces derivable from potentials (figure 2). It is also assumed that the fluids are at the same temperature and that fluid 2 is completely immersed in fluid 1 and does not make contact with the wall 3. The sum U of the surface tension energy and the potential energy of this system is given by:

$$U = \int_{S_{12}} \sigma \, dS + \int_{V_1} P_1 \, dV + \int_{V_2} P_2 \, dV \qquad [2]$$

in which S_{12} is the contact surface between the fluids, V_1 and V_2 the regions occupied by each and P_1 and P_2 their potential energies per unit volume.

For such a system one can show that the principle of minimum free energy as stated by Gibbs (1876, 1878) for systems in which surface tension is important reduces to the principle that in order for the system to be in stable equilibrium it is necessary that the interface S_{12} assume such a shape that U reaches a minimum value.

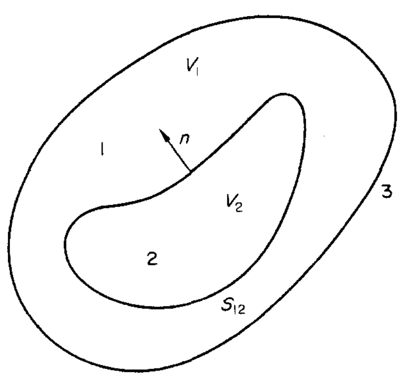

Figure 2. General system of two fluids in contact.

The two extreme cases of this principle are well known. If surface tension is negligible with respect to potential forces, it is well known that a stable configuration is obtained if the potential energy of the system attains a minimum. For example if a container is filled with two fluids of different density and if gravity is the only important force, then equilibrium is reached when the contact surface between the two fluids is horizontal (the potential energy reaching an extremum). This equilibrium is stable only if the lower fluid is the more dense one, i.e. the potential energy of the system is minimal.

On the other hand if only surface tension forces are important it is well known that the contact surface will have a stable equilibrium shape if its area is minimal (minimal surface tension energy). It is for this reason that a volume of fluid immersed in another fluid with which it does not mix will assume a spherical shape if the fluids have the same density. The spherical shape is the one with the smallest surface area for the same volume.

If both surface tension and potential forces are present, it is not surprising therefore to see that the sum of potential energy and surface tension energy has to be a minimum. Several interesting applications of this principle can be found in Landau & Lifschitz (1959).

The minimum principle stated above can be expressed mathematically by requiring that the first and the second variations of the sum of surface tension and potential energy of the system with respect to the value reached at a stable equilibrium should satisfy the conditions:

$$\delta U = 0 \qquad [3]$$

and

$$\delta^2 U > 0. \qquad [4]$$

The first condition expresses the fact that the total energy reaches an extremum, or in other words that the forces acting upon the interface are in equilibrium. The second condition certifies that the extremum is a minimum and therefore that the equilibrium is stable.

The first and the second variations of the potential energy are due to the displacements $\delta \mathbf{x}(\mathbf{x})$ of the volume elements. Since the two fluids are incompressible it is necessary that:

$$\operatorname{div}(\delta \mathbf{x}) = 0 \qquad [5]$$

in which $\mathbf{x}$ is the position vector.

As shown by Berghmans (1972) the first variation can be written as:

$$\delta U = \int_{S_{12}} \left[p_1 - p_2 - \sigma\left(\frac{1}{R_k} + \frac{1}{R_l}\right)\right] N \, \mathrm{d}S \qquad [6]$$

in which

$$N = \delta \mathbf{x} \cdot \mathbf{n}, \qquad [7]$$

while p_1 and p_2 are the pressures on each side of the interface of which R_k and R_l are the principal radii of curvature and $\mathbf{n}$ the unit normal vector directed into region 1. In order for condition [3] to be satisfied for any displacement N of the interface, it is necessary that an equilibrium shape satisfy:

$$p_1 - p_2 = \sigma\left(\frac{1}{R_k} + \frac{1}{R_l}\right) \quad \text{on } S_{12}. \qquad [8]$$

This is Laplace's equation for capillary phenomena which expresses the equilibrium of the interface.

From [6] the second variation of U can be written as:

$$\delta^2 U = \int_{S_{12}} \left\{ N\delta\left[p_1 - p_2 - \sigma\left(\frac{1}{R_k} + \frac{1}{R_l}\right)\right] + \left[p_1 - p_2 - \sigma\left(\frac{1}{R_k} + \frac{1}{R_l}\right)\right] \delta N \right\} \mathrm{d}S. \qquad [9]$$

Since the variation is taken around an equilibrium shape for which [8] is valid, [9] reduces to:

$$\delta^2 U = \int_{S_{12}} N\delta \left[p_1 - p_2 - \sigma \left(\frac{1}{R_k} + \frac{1}{R_l} \right) \right] \mathrm{d}S. \quad [10]$$

The term in the square brackets varies only in a direction perpendicular to the interface, as follows from [8], therefore:

$$\delta \left[p_1 - p_2 - \sigma \left(\frac{1}{R_k} + \frac{1}{R_l} \right) \right] = N \frac{\partial}{\partial n} \left[p_1 - p_2 - \sigma \left(\frac{1}{R_k} + \frac{1}{R_l} \right) \right] \quad [11]$$

in which $\partial/\partial n$ designates partial differentiation in the direction normal to the interface.

Applying the formula for the variation of the average curvature of the surface S_{12}, as given by Blaschke (1930), yields:

$$\delta^2 U = \sigma \int_{S_{12}} (\tau N - \nabla_S^2 N) N \, \mathrm{d}S \quad [12]$$

in which ∇_S^2 is the surface Laplacian on S_{12} and:

$$\tau = \frac{1}{\sigma} \frac{\partial (p_1 - p_2)}{\partial n} - \left(\frac{1}{R_k} + \frac{1}{R_l} \right)^2 + \frac{2}{R_k R_l}. \quad [13]$$

In this last expression the first term represents the contribution of the potential energy to the second variation of U. The pressure difference is determined by integrating the momentum equations for both fluids. The last two terms of [13] give a contribution of the surface tension energy to the second variation.

To determine stability, it is the sign of the right hand side of [12] which is important. By means of the theory of the equivalence between differential equations and variational problems, as developed in the work of Courant & Hilbert (1968), it can be shown (Berghmans 1972) that finding the smallest value of the right hand side corresponds to finding the smallest eigenvalue ν of a differential equation. In particular this theory leads to the expression:

$$\delta^2 U = \nu\sigma \int_{S_{12}} N^2 \mathrm{d}S \quad [14]$$

in which ν is the eigenvalue of the differential equation:

$$\tau N - \nabla_S^2 N - \nu N = 0. \quad [15]$$

Thus it may be said that the stability problem has been reduced to an eigenvalue problem. It suffices to look for equilibrium interfaces, i.e. interfaces satisfying [8] and then to determine whether the differential equation [15] has negative eigenvalues or not. This differential equation is not easy to solve in general since the coefficient τ depends upon the shape of the interface. However in the case of a cylindrical interface the solution can be readily obtained.

Consider a cylindrical column of liquid of radius R in contact with another fluid. For a cylinder one of the principle radii of curvature is infinite, the other being equal to the radius of the cylinder. If gravity can be neglected, one obtains then from [13]:

$$\tau = -\frac{1}{R^2}. \quad [16]$$

Furthermore for a cylinder the surface Laplacian is:

$$\nabla_S^2 = \frac{1}{R^2}\frac{\partial^2}{\partial\theta^2} + \frac{\partial^2}{\partial z^2}. \qquad [17]$$

If the perturbation N is independent of the azimuthal angle θ, while varying sinusoidally ($\sin k_z z$) in the axial direction z, then:

$$\nabla_S^2 N = -k_z^2 N. \qquad [18]$$

Therefore from [16], [18] and [15] it follows that:

$$\left(-\frac{1}{R^2} + k_z^2 - \nu\right) N = 0 \qquad [19]$$

or

$$\nu = \frac{-1}{R^2} + k_z^2. \qquad [20]$$

Since the wavenumber k_z is related to the wavelength λ by:

$$k_z = \frac{2\pi}{\lambda} \qquad [21]$$

it follows that positive values of ν, or stability, will be obtained for displacements with wavelength $\lambda < \lambda_c$ in which the critical wavelength λ_c is given by:

$$\lambda_c = 2\pi R. \qquad [22]$$

This is the classical result as mentioned by Lamb (1932). Considerable complication of the problem occurs when gravity is taken into account, as will be shown later.

Thus far it is possible to determine which interfaces are stable and which are not. Yet if a perturbation for which the interface is unstable is applied, it is not possible to determine how fast this perturbation will grow in time. Therefore the most unstable wavelength cannot be determined by the above variational approach. However if the perturbations of the interface are assumed to vary exponentially in time t:

$$\delta\mathbf{x}(\mathbf{x}, t) = \delta\mathbf{x}(\mathbf{x})e^{\omega t} \qquad [23]$$

then the expression for $\delta^2 U$ remains the same except that the normal displacement N now will have the same time dependence as $\delta\mathbf{x}(\mathbf{x}, t)$.

The system under consideration consists of a rigid container filled with two fluids which are stagnant at equilibrium. The total energy of this system remains constant since no energy can be transported through the container walls. Yet when considering time-dependent perturbations of the equilibrium state, part of this energy will appear in the form of kinetic energy (K). This is not in contradiction with the assumption of stagnant fluids since this refers to the equilibrium state (where $K = 0$). At all times it is true that:

$$K + U = \text{constant}$$

or

$$\delta K + \delta U = 0 \qquad [24]$$

and

$$\delta^2 K + \delta^2 U = 0. \qquad [25]$$

Condition [24] is always satisfied since the perturbation is around an equilibrium interface such that [3] is satisfied and because of the fluids being stagnant in the steady state so that the variation of the kinetic energy is of the second order in the perturbation. Substituting [14] into [25], it follows that:

$$\delta^2 K = -\nu\sigma \int_{S_{12}} N^2 \, \mathrm{d}S. \tag{26}$$

The second variation of the kinetic energy of the system is related to the velocity of the displacements $\mathbf{v}$ by:

$$\delta^2 K = \frac{1}{2}\rho_1 \int_{V_1} v^2 \, \mathrm{d}V + \frac{1}{2}\rho_2 \int_{V_2} v^2 \, \mathrm{d}V, \tag{27}$$

while the velocities are related to the displacements by:

$$\mathbf{v} = \frac{\partial \delta \mathbf{x}(\mathbf{x}, t)}{\partial t} = \omega \delta \mathbf{x}(\mathbf{x}, t). \tag{28}$$

Substituting this last expression into [27] gives:

$$\delta^2 K = \frac{1}{2}\rho_1 \omega^2 \int_{V_1} \delta\mathbf{x} \cdot \delta\mathbf{x} \, \mathrm{d}V + \frac{1}{2}\rho_2 \omega^2 \int_{V_2} \delta\mathbf{x} \cdot \delta\mathbf{x} \, \mathrm{d}V. \tag{29}$$

Considering the momentum equation for each fluid and retaining only terms of first order in the perturbations, one has for each fluid:

$$\rho \frac{\partial \mathbf{v}}{\partial t} = -\operatorname{grad} p. \tag{30}$$

This shows that the velocity perturbations can be derived from a potential. Together with [28] it is therefore possible to put:

$$\delta \mathbf{x}(\mathbf{x}) = \operatorname{grad} \phi. \tag{31}$$

Substituting this expression into [29] one finds, after partial integration:

$$\delta^2 K = \frac{1}{2}\omega^2 \int_{S_{12}} (\rho_2\phi_2 - \rho_1\phi_1) N \, \mathrm{d}S. \tag{32}$$

Finally from [26] and [32] it follows that:

$$\frac{1}{2}\omega^2 \int_{S_{12}} (\rho_2\phi_2 - \rho_1\phi_1) N \, \mathrm{d}S = -\nu\sigma \int_{S_{12}} N^2 \, \mathrm{d}S. \tag{33}$$

This relation can be used to calculate the growth rate of the displacements of the interface. The potentials have to satisfy the conditions:

$$\nabla^2 \phi_{1,2} = 0 \tag{34}$$

and

$$\frac{\partial \phi_1}{\partial n} = N = \frac{\partial \phi_2}{\partial n} \quad \text{at } S_{12}. \tag{35}$$

Condition [34] follows from the incompressibility of the two fluids as expressed by [5]. The boundary condition [35] follows from [7] and [31].

In conclusion one can say that to study the stability of an equilibrium interface one has to look for the existence of solutions of the eigenvalue problem [15] with negative eigenvalues. If such solutions exist, their growth rate ω can be determined from [33]. Unstable perturbations will give rise to positive values of ω^2 showing that the perturbations grow exponentially in time. Stable perturbations give rise to negative values of ω^2, indicating oscillatory behaviour in time. The "most dangerous" wavelength is the one which grows fastest in time and therefore is the wavelength of the unstable perturbation with the largest ω.

In principle this procedure can be applied to any equilibrium interface. In practice finding a closed form solution to the eigenvalue problem is not always possible.

STABILITY IN A GRAVITATIONAL FORCE FIELD

It will be assumed that the variational approach to interfacial stability problems as developed above can be applied to study the stability of the vapor–liquid interface around a heated horizontal cylinder in a normal gravitational field.

Several conditions have to be satisfied in order to be able to apply the principle of minimum total energy. First the fluids have to be stagnant in the steady state. This can be readily assumed since the effects of natural convection and induced motion are negligible, as mentioned by Lienhard & Wong (1964). Incompressibility and immiscibility are also satisfied in the experiments under consideration. The only volume forces are gravitational forces which can be derived from a potential. Furthermore both fluids are assumed to be inviscid. It has been shown by Dhir (1972) that the effect of liquid viscosity μ_f can be neglected if:

$$\frac{\rho_f \sigma^{3/4}}{\mu_f g^{1/4}(\rho_f - \rho_g)^{3/4}} \gg 1$$

which is satisfied in the normal gravity experiments of Lienhard & Sun (1970). Finally it was assumed that both fluids are at the same temperature. In the experiments the liquid is uniformly heated to a temperature very close to the boiling temperature by auxiliary heating elements, before the horizontal cylinder is heated. Effects of temperature gradients in the liquid are therefore negligible. A temperature gradient does exist in the vapor film. However due to the small density of the vapor, any effects of this gradient can be expected to be small. The only way then in which the heat flux can influence the experiments is through the mass transfer at the interface, caused by evaporation. It has been shown by Moureau (1972) that this mass transfer can be neglected if:

$$\frac{q_c q_t}{h_{fg}^2 \rho_g \rho_f s g} \ll 1,$$

in which h_{fg} is the heat of evaporation while q_c and q_t are the conduction heat flux and the total heat flux (conduction and radiation) respectively, through the vapor film of thickness s. In reality the condition above means that one should not approach the critical conditions. Also this is satisfied by the experiments of Lienhard & Sun (1970). In conclusion it can be said therefore that the experiments under consideration satisfy all the conditions necessary to allow application of the variational approach.

If the heated liquid is taken as fluid 1 and the vapor as fluid 2, then fluid 2 is in contact with a solid wall (the heated cylinder) which it covers completely, but not with the container wall. Therefore the displacement of fluid 2 has to be zero at the cylinder, which represents an additional boundary condition to ϕ_2. However from [35] it follows that ϕ_1 and ϕ_2 are of the same order of magnitude. In the present experiments:

$$\rho_2 \ll \rho_1. \qquad [36]$$

This allows the term with ρ_2 in [33] to be neglected such that therefore ϕ_2 does not have to be determined. This renders the above boundary condition and the presence of the cylinder wall completely irrelevant. The container wall 3 is taken to be very far from the cylinder such that there the boundary condition on fluid 1 is that its displacement becomes vanishingly small.

To study the stability of a cylindrical interface use will be made of a cylindrical coordinate system (figure 3). The interface has a radius R which is approximately equal to the radius of the

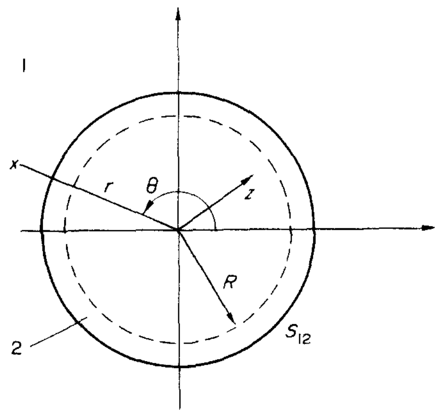

Figure 3. Coordinate system used to study the stability of a cylindrical vapor film in a normal gravitational field.

heater. If a perturbation of the interface is considered which has an angular dependence then the surface Laplacian is given by [17]. The coefficient τ of the eigenvalue equation [15] becomes:

$$\tau = \frac{1}{\sigma}\frac{\partial(p_1 - p_2)}{\partial r} - \frac{1}{R^2}, \quad [37]$$

because one of the principal radii of curvature is R and the other is infinite. The eigenvalue problem can then be written as:

$$\left[\frac{1}{\sigma}\frac{\partial(p_1 - p_2)}{\partial r} - \frac{1}{R^2}\right]N - \frac{1}{R^2}\frac{\partial^2 N}{\partial\theta^2} - \frac{\partial^2 N}{\partial z^2} - \nu N = 0. \quad [38]$$

For a gravitational force field the steady state pressure distribution across the interface is:

$$p_1 - p_2 = (\rho_2 - \rho_1)gr\sin\theta + \text{constant}, \quad [39]$$

so that:

$$\frac{\partial(p_1 - p_2)}{\partial r} = (\rho_2 - \rho_1)g\sin\theta. \quad [40]$$

The constant term in [39] contains the surface tension pressure difference σ/R. Yet this does not enter into the picture since $[\partial(p_1 - p_2)/\partial r]$ refers to the variation of the pressures at equilibrium as it represents the contribution of the change in potential energy to the change in U.

With [40] the eigenvalue problem [38] then becomes:

$$\frac{1}{R^2}\frac{\partial^2 N}{\partial\theta^2} + \frac{\partial^2 N}{\partial z^2} + \left[\frac{(\rho_1 - \rho_2)g}{\sigma}\sin\theta + \frac{1}{R^2} + \nu\right]N = 0. \quad [41]$$

The displacement of the interface will be assumed to vary sinusoidally in the z-direction:

$$N(\theta, z) = F(\theta)\sin k_z z \quad [42]$$

in which the wave number k_z is related to the wavelength λ by [21]. Substituting $\eta = \theta/2 + \pi/4$ into [41] yields:

$$\frac{d^2F}{d\eta^2} + (a - 2q\cos 2\eta)F = 0, \qquad [43]$$

with

$$a = 4 + 4\nu R^2 - 4k_z^2R^2 \quad \text{and} \quad q = 2B^2, \qquad [44]$$

where the Bond number is defined by:

$$B^2 = \frac{(\rho_1 - \rho_2)gR^2}{\sigma}. \qquad [45]$$

This dimensionless group represents the ratio of gravity forces and surface tension forces.

Equation [43] is the well-known Mathieu equation. This equation has an infinite number of periodic solutions each being characterized by a specific relation between the coefficients a and q. For small values of q (or B) these solutions are listed in Abramowitz & Stegun (1968). The solutions one is interested in here are the ones which give rise to negative values of ν. For small B only two such solutions exist, all the others therefore being stable solutions. These unstable solutions F_α and F_β are:

$$F_\alpha = F_{0,\alpha}\left[\left(1 - \frac{B^4}{32} + \frac{B^6}{64}\right)\sin\left(\frac{\theta}{2} + \frac{\pi}{4}\right) + \left(-\frac{B^2}{4} + \frac{B^4}{16}\right)\sin\left(\frac{3\theta}{2} + \frac{3\pi}{4}\right)\right], \qquad [46]$$

for which the relation between a and q yields:

$$a_\alpha = 1 - 2B^2 - \frac{B^4}{2} + \frac{B^6}{8}, \qquad [47]$$

and

$$F_\beta = F_{0,\beta}\left[1 - \frac{B^4}{4} + \left(B^2 - \frac{11}{16}B^6\right)\sin\theta\right], \qquad [48]$$

while a and B are related by:

$$a_\beta = -2B^4 + \tfrac{7}{8}B^8. \qquad [49]$$

The coefficients $F_{0,\alpha}$ and $F_{0,\beta}$ are much smaller than R since the perturbations are assumed small in this linearized theory. The solutions for F_α and F_β are all valid for small values of B only ($B < 0.8$). The series expansions are truncated such that the contribution of the neglected terms is less than 0.10 for $B < 0.8$. For larger values of B, more terms have to be retained in the series expansions of F_α and F_β due to the slow convergence of these series. The results obtained here (up to [71]) are valid only for $B < 0.8$.

It should be noted that the first solution F_α satisfies the condition:

$$F_\alpha\left(-\frac{\pi}{2}\right) = 0, \qquad [50]$$

and is therefore of the asymmetrical type, with zero displacement at the bottom of the interface. The second solution F_β reduces to the axi-symmetric solution, discussed in the previous section, for $B = 0$.

The relation between the eigenvalue ν and the Bond number is found by substituting a_α

respectively a_β from [47], respectively [49] into [44] yielding:

$$\nu_\alpha R^2 = -\frac{3}{4} - \frac{B^2}{2} - \frac{B^4}{8} + \frac{B^6}{32} + k_z^2 R^2, \qquad [51]$$

$$\nu_\beta R^2 = -1 - \frac{B^4}{2} + \frac{7}{32} B^8 + k_z^2 R^2. \qquad [52]$$

As shown above the transition from stable to unstable perturbations is characterized by a zero value of ν. From [51] resp. [52] it follows that this transition occurs at critical values of the wavelength for the F_α and F_β solutions given by:

$$\lambda_{c,\alpha} = \frac{2\pi R}{\left(\frac{3}{4} + \frac{B^2}{2} + \frac{B^4}{8} - \frac{B^6}{32}\right)^{1/2}}, \qquad [53]$$

$$\lambda_{c,\beta} = \frac{2\pi R}{\left(1 + \frac{B^4}{2} - \frac{7}{32} B^8\right)^{1/2}}. \qquad [54]$$

The latter reduces to [22] if gravitational effects are neglected. Perturbations of the $F_\alpha (F_\beta)$ type are unstable if their wavelength is larger than $\lambda_{c,\alpha} (\lambda_{c,\beta})$. To determine the growth rate of these perturbations the corresponding potentials $\phi_{1,\alpha}$ and $\phi_{1,\beta}$ satisfying [34] and [35] have to be found. Separation of variables applied to the Laplace equation yields:

$$\phi_{1,\alpha}(r, \theta, z) = C_\alpha \sin\left(\frac{\theta}{2} + \frac{\pi}{4}\right) \frac{K_{1/2}(k_z r)}{\left.\frac{dK_{1/2}}{dr}\right|_{r=R}} \sin k_z z + D_\alpha \sin\left(\frac{3\theta}{2} + \frac{3\pi}{4}\right) \frac{K_{3/2}(k_z r)}{\left.\frac{dK_{3/2}}{dr}\right|_{r=R}} \sin k_z z, \qquad [55]$$

with

$$C_\alpha = F_{0,\alpha}\left(1 - \frac{B^4}{32} + \frac{B^6}{64}\right) \quad \text{and} \quad D_\alpha = F_{0,\alpha}\left(-\frac{B^2}{4} + \frac{B^4}{16}\right), \qquad [56]$$

$$\phi_{1,\beta}(r, \theta, z) = C_\beta \frac{K_0(k_z r)}{\left.\frac{dK_0}{dr}\right|_{r=R}} \sin k_z z + D_\beta \sin\theta \frac{K_1(k_z r)}{\left.\frac{dK_1}{dr}\right|_{r=R}} \sin k_z z, \qquad [57]$$

with

$$C_\beta = F_{0,\beta}\left(1 - \frac{B^4}{4}\right) \quad \text{and} \quad D_\beta = F_{0,\beta}\left(B^2 - \frac{11}{16} B^6\right), \qquad [58]$$

in which K_0, $K_{1/2}$, K_1, $K_{3/2}$, are the modified Bessel functions of orders 0, $\frac{1}{2}$, 1, $\frac{3}{2}$ respectively.

These potentials have to be substituted into [33], which for a cylindrical interface becomes:

$$-\frac{1}{2}\rho_1\omega^2 \int_0^{2\pi}\int_0^\lambda R\phi_1 N \, dz \, d\theta = -\nu\sigma \int_0^{2\pi}\int_0^\lambda RN^2 \, dz \, d\theta. \qquad [59]$$

Furthermore ν_α and ν_β have to be substituted from [51] and [52].

This yields for the F_α solution:

$$\frac{\frac{1}{2}\rho_1\omega_\alpha^2 R^3}{\sigma} = \left(\frac{3}{4} + \frac{B^2}{2} + \frac{B^4}{8} - \frac{B^6}{32} - k_z^2 R^2\right) \frac{C_\alpha^2 + D_\alpha^2}{\frac{C_\alpha^2}{k_z R + \frac{1}{2}} + \frac{D_\alpha^2 (k_z R + 1)}{(k_z R + \frac{1}{2})(k_z R + 1) + 1}}, \qquad [60]$$

in which use was made of the relations:

$$\frac{k_z R \left.\frac{dK_{1/2}}{d(k_z r)}\right|_{r=R}}{K_{1/2}(k_z R)} = -\left(\frac{1}{2} + k_z R\right) \quad \text{and} \quad \frac{k_z R \left.\frac{dK_{3/2}}{d(k_z r)}\right|_{r=R}}{K_{3/2}(k_z R)} = -\frac{\left(k_z R + \frac{1}{2}\right)(k_z R + 1) + 1}{k_z R + 1}. \qquad [61]$$

For $B < 0.8$, D_α can be neglected such that:

$$\frac{\frac{1}{2}\rho_1\omega_\alpha^2R^3}{\sigma} = \left(\frac{3}{4}+\frac{B^2}{2}+\frac{B^4}{8}-\frac{B^6}{32}-k_z^2R^2\right)\left(k_zR+\frac{1}{2}\right). \qquad [62]$$

This equation shows the dependence of the growth rate ω_α upon the wavelength. The maximum growth rate is reached at the "most dangerous" wavelength given by:

$$\lambda_{d,\alpha} = \frac{12\pi R}{\left(10+6B^2+\frac{3B^4}{2}-\frac{3B^6}{8}\right)^{1/2}-1}. \qquad [63]$$

Substituting [57] and [58] into [59] one obtains for the F_β perturbation:

$$\frac{\frac{1}{2}\rho_1\omega_\beta^2R^3}{\sigma} = \left(1+\frac{B^4}{2}-\frac{7}{32}B^8-k_z^2R^2\right)\frac{2C_\beta^2+D_\beta^2}{2C_\beta^2\frac{K_0(k_zR)}{k_zRK_1(k_zR)}-D_\beta^2\frac{K_1(k_zR)}{[\mathrm{d}K_1/\mathrm{d}(k_zr)]|_{r=R}}}, \qquad [64]$$

in which use was made of:

$$\frac{\mathrm{d}K_0(k_zr)}{\mathrm{d}(k_zr)} = -k_zrK_1(k_zr). \qquad [65]$$

The dependence of ω_β upon k_z is much more complicated here yet here too the growth rate reaches a maximum at a value of k_zR which depends upon B.

The "most dangerous" wavelengths calculated from the maxima of ω_α and ω_β, as given by [60] and [64], are plotted on figure 4 in a dimensionless form by defining:

$$\Lambda = = \frac{\lambda_d}{\lambda_{dF}} \qquad [66]$$

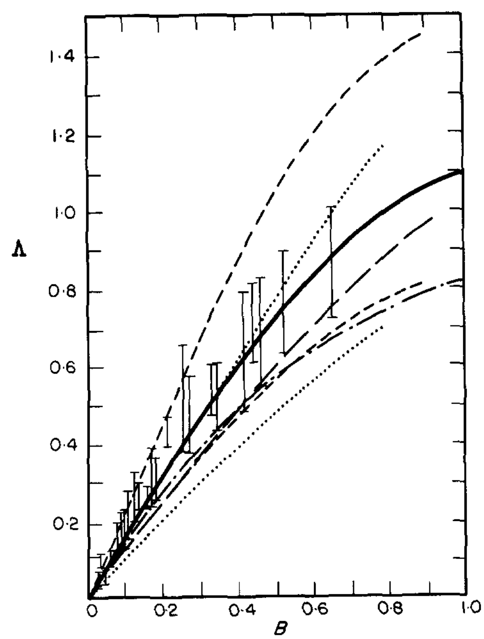

Figure 4. Most dangerous wavelength of a cylindrical vapor film in a normal gravitational field. I Experimental results of Lienhard & Sun (1970); —·— Λ at maximal growth rate as found by Lienhard & Wong (1964); —— Λ at maximal growth rate for the F_α perturbation; — — Λ at maximal growth rate for the F_β perturbation; ---- Λ at 98 per cent of the maximal growth rate for the F_α perturbation; ······ Λ at 98 per cent of the maximal growth rate for the F_β perturbation.

with

$$\lambda_{dF} = \frac{2\beta^{1/2}\pi}{\left(\frac{\rho_1 g}{\sigma}\right)^{1/2}} \quad [67]$$

in which λ_{dF} is the "most dangerous" wavelength for a flat plate. On the same graph the experimental observations of Lienhard & Sun (1970), for normal gravity, are plotted. Considering the large spread of experimental data, the values of Λ for which the growth rate is 2% less than the maximum growth rate (for the same B) are also plotted. On figure 5 the dimensionless growth rate Ω defined by:

$$\Omega = \frac{\omega}{\left[\frac{\sigma}{\rho_1 R^3}\right]^{1/2}} \quad [68]$$

is plotted vs λ/R for the F_α perturbation and for $B = 0$. The same qualitative behavior is obtained for $B > 0$. This figure illustrates how very large changes of λ with respect to the "most dangerous" one give rise to small changes in the growth rate. The instability is not very selective and a broad wavelength band can contribute to instability. This explains the large spread in the experimental data.

It can be concluded from figure 4 that the F_β perturbations are not responsible for the observed instabilities since their "most dangerous" wavelengths are too small, also if a 2% band in growth rate is allowed. The F_α perturbations however explain the data perfectly if the effect of the finite thickness of the vapor film is taken into account, effect which is most pronounced at small radii ($B < 0.3$). It should also be noticed that the observations fall completely within the wavelength band based upon a 3% variation from the maximum growth rate, even if the film thickness is not considered.

The analysis of Lienhard & Wong (1964) leads to the dimensionless "most dangerous" wavelength given by:

$$\Lambda_{LW} = \left(\frac{B^2}{B^2 + \frac{1}{2}}\right)^{1/2}, \quad [69]$$

and the growth rate given by:

$$\frac{\frac{1}{2}\rho_1 \omega_{LW}^2 R^3}{\sigma} = \frac{k_z R}{2}\left(\frac{1}{2} + B^2 - k_z^2 R^2\right). \quad [70]$$

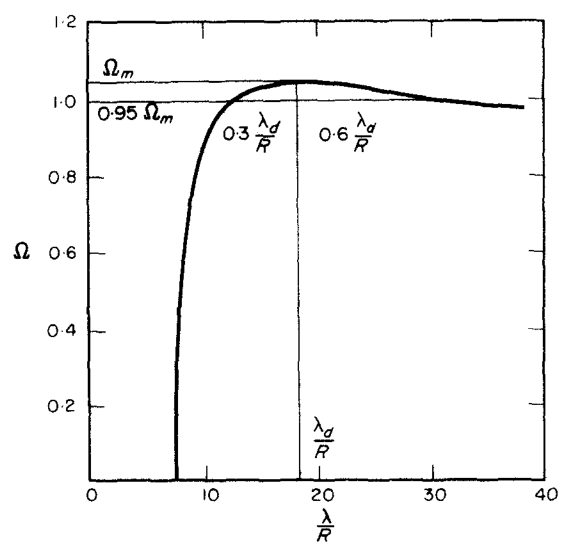

Figure 5. Growth rate Ω as function of λ/R for the F_α perturbation at $B = 0$.

Lienhard & Sun (1970) contended that with a wavelength band due to a 10% change in ω it is possible to cover the data. However it should be noticed that their "most dangerous" wavelength is somewhat low. Indeed, regardless of the magnitude of the spread in ω the observed wavelengths should be located on both sides ("straddle") of the "most dangerous" wavelength, since the latter is the one corresponding to the fastest growing disturbance and therefore is the one most likely to be observed. Yet figure 4 shows that Λ_{LW} is always somewhat under the observations.

It can be concluded from this that the F_α-type perturbations are the only ones which explain the data well. From [63] and [66] it follows then that the most unstable wavelength for $B < 0.8$ is given by:

$$\Lambda_\alpha = \frac{2\,3^{1/2}B}{\left[10+6B^2+\frac{3B^4}{2}-\frac{3B^6}{8}\right]^{1/2}-1}. \qquad [71]$$

The F_α solution of the differential equation [43] is given by [46] for small B. The right-hand side of [46] contains the first two terms of a series expansion in $\sin[(2n+1)(\theta/2+\pi/4)]$. For $B > 1$ more terms have to be considered in this expansion and it becomes very difficult to obtain a closed-form expression for Λ_α. In addition, for large B, other solutions than F_α and F_β, also giving rise to negative values of ν become possible, which further complicates the matter. However also for large B the eigenvalue equation [41] is valid. Starting from this equation, a derivation now follows in which the limit value of Λ for large B is determined.

The eigenvalue equation [41] can also be written as:

$$-R^2\left[\frac{1}{N}\frac{\partial^2 N}{(R\theta)^2}+\frac{1}{N}\frac{\partial^2 N}{\partial z^2}\right] = B^2\sin\theta + 1 + \nu R^2. \qquad [72]$$

Considering that the terms in the square brackets are proportional to the inverse square of the wavelengths in the θ and z directions, [72] shows that for large B the wavelengths are much smaller than R. This justifies a simplified approach in which a rectangular coordinate system x, y, z, located at the top of the cylinder is used. For distances from the top of the cylinder which are of the order of a wavelength it is possible to put:

$$\sin\theta = 1 \quad \text{and} \quad \frac{\partial^2}{\partial(R\theta)^2} = \frac{\partial^2}{\partial x^2}$$

such that for large B, [72] becomes:

$$\frac{\partial^2 N}{\partial x^2}+\frac{\partial^2 N}{\partial z^2} = -\frac{1}{R^2}(B^2+\nu R^2)N. \qquad [73]$$

The range of Bond numbers for which this flat plate approach is valid can be estimated as follows. The significant wavelength here is the flat plate wavelength λ_{dF} as defined by [67]. It is assumed that this wavelength is smaller than the radius of the cylinder:

$$\lambda_{dF} < R,$$

or by making use of [67]:

$$\frac{2\,3^{1/2}\pi}{\left(\frac{\rho_1 g}{\sigma}\right)^{1/2}} < R.$$

This yields:

$$B > 10.$$

Thus it is found that the above approach in which only the region close to the upper part of the cylinder is considered is valid only for $B > 10$.

If the perturbations are assumed to have a sinusoidal behavior in both x and z directions (the y-axis being the vertical direction):

$$N = N_0 \sin k_x x \sin k_z z, \qquad [74]$$

then [73] gives:

$$\nu R^2 = (k_x^2 + k_z^2)R^2 - B^2.$$

Solving Laplaces equation [34] in rectangular coordinates with [35] as boundary condition and substitution into [33] gives:

$$\omega^2 = \frac{g}{R}\left(kR - \frac{k^3R^3}{B^2}\right), \qquad [75]$$

with

$$k^2 = k_x^2 + k_z^2. \qquad [76]$$

Maximal growth rate is obtained when:

$$kR = \frac{B}{3^{1/2}},$$

or for $k_x = k_z$:

$$k_z R = \frac{B}{6^{1/2}}, \qquad [77]$$

so that the "most dangerous" wavelength is given by:

$$\lambda_d = \frac{2\, 6^{1/2} \pi R}{B}, \qquad [78]$$

or in terms of the dimensionless "most dangerous" wavelength:

$$\Lambda = 2^{1/2} \quad \text{for } B > 10. \qquad [79]$$

This is the same result as obtained by Sernas (1969) in his three-dimensional analysis of the stability of a vapor film on a horizontal flat plate heater.

In conclusion of this section it can be said that for small B the variational approach leads to results which are in excellent agreement with experiment. In addition, for large B, the variational approach gives the same result as the one obtained by Sernas (1969).

STABILITY IN A CENTRIFUGAL FORCE FIELD

Lienhard & Sun (1970) published considerable data from experiments in which a cylindrical heater is placed in a centrifuge. This allowed to vary the volume forces acting on the fluid and thus to enlarge the domain of investigation. They compared these results with the predictions of the two-dimensional analysis of Lienhard & Wong (1964) and obtained rather good agreement. In this section it will be shown that the latter theory is applicable to a rotating system only under certain conditions.

The major distinction between a system under the influence of gravity and one in a centrifugal force field lies in the fact that in the latter the fluids are not stagnant, their rotation giving rise to centrifugal and Coriolis forces. The latter forces give rise to first order perturbations in the momentum equation so that the stagnant fluid approach as taken by Lienhard & Wong (1964) is not valid any more. Lienhard & Sun (1970) assumed that this analysis made for a normal gravitational force field remains valid in a centrifugal force field by replacing the gravitational constant by the centrifugal acceleration. Similarly the variational analysis given above is applicable only to fluids which are stagnant with respect to some inertial reference frame so that it cannot be applied to a rotating system. Therefore a new perturbation analysis will be developed here.

In the centrifuge experiments reported by Lienhard & Sun (1970), the heated wire is placed in a rigid container filled with liquid (see figure 6). The wire is located a distance L from the axis of rotation in a plane perpendicular to this axis while it forms a right angle with the connecting rod OH. The axis of rotation will be taken as the z-axis of a cylindrical coordinate system r, θ, z. This system rotates together with the fluid at a rate γ. For such a coordinate system the momentum equation for an incompressible, inviscid fluid becomes (Chandrasekhar 1961):

$$\rho\frac{\partial \mathbf{v}}{\partial t}+\rho(\mathbf{v}\cdot\text{grad})\mathbf{v} = -\text{grad}\, p + \rho[\text{grad}\,\tfrac{1}{2}(\gamma\times\mathbf{r})^2] - 2\rho\gamma\times\mathbf{v}. \qquad [80]$$

The second and the third term on the right hand side of this equation represent the centrifugal force and the Coriolis force respectively. Thus varying γ allows for variation of the gravitational constant g. Considering only terms which are of the first order in the perturbations, [80] reduces to:

$$\rho\frac{\partial \mathbf{v}}{\partial t}+2\rho\gamma\times\mathbf{v} = -\text{grad}\, p. \qquad [81]$$

Comparing [81] to the first-order momentum equation for stagnant fluids as given by [30], it is seen that Coriolis forces come into play. The ratio Γ of the inertia force and the Coriolis force is given by:

$$\Gamma = \frac{\omega}{2\gamma} \qquad [82]$$

if it is assumed that the velocity varies as $e^{\omega t}$ which is the case during the initial stage of growth of an unstable perturbation. Substituting the maximum value of ω_α from [62] and [63] into [82], it is found that for small B:

$$\Gamma = \frac{1}{2\gamma}\left[\frac{\sigma}{\rho_1 R^3}\right]^{1/2} = \frac{1}{2B}\left[\frac{L}{R}\right]^{1/2}. \qquad [83]$$

In which use is made of the Bond number defined as:

$$B = \left[\frac{\rho_1\gamma^2 LR}{\sigma}\right]^{1/2}.$$

In the experiments of Lienhard & Sun (1970) the value of L/R is of order 10^4. Thus [83] seems to

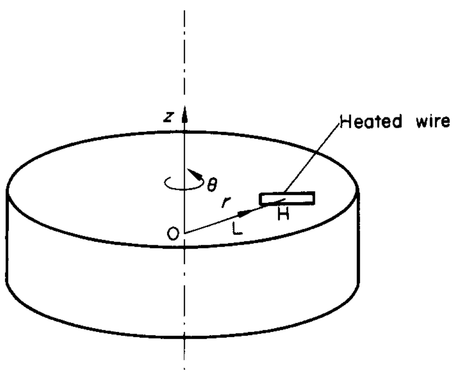

Figure 6. Coordinate system used to study the stability of a cylindrical vapor film in a centrifugal force field.

indicate that the Coriolis forces do not play a dominant role, unless B is very large. Yet as shown in figure 4, small variations in ω (say $\pm 2\%$) give rise to large variations in $\Lambda(\pm 10\%)$. Therefore even if Coriolis forces give only a $\pm 1\%$ variation in ω (i.e. $\Gamma = 100$) then a variation of Λ of $\pm 5\%$ may be expected. Considering [83] it may be said therefore that already for $B > 0.5$, the effect of Coriolis forces may give rise to appreciable changes in the Λ, B curve.

This agrees with the experimental data. As shown on figure 7, the data reported by Lienhard & Sun (1970) tends to show an increased spread of the centrifuge data for $B > 0.5$. Furthermore this figure indicates that the values of Λ obtained in a centrifuge are somewhat smaller than the ones obtained in a normal gravitational field for $B > 0.5$. As shown above this different behavior may be due to the effect of Coriolis forces. Unfortunately a complete theory taking this effect into account is rather complicated.

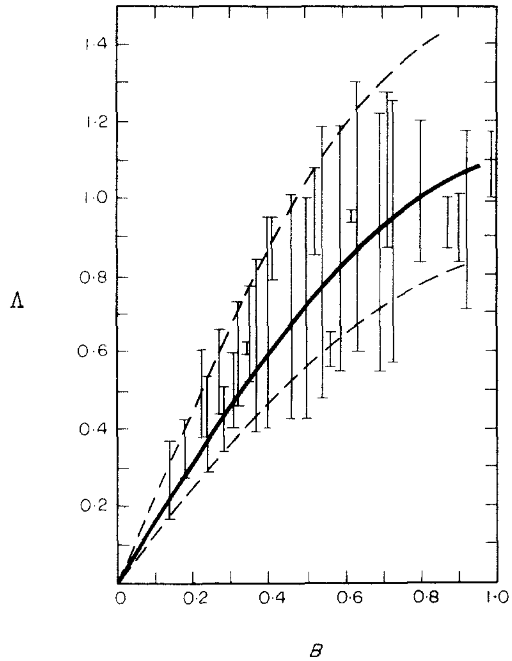

Figure 7. Most dangerous wavelength of a cylindrical vapor film in a centrifugal force field. I Experimental results of Lienhard & Sun (1970); —— Λ at maximal growth rate from present theory; ---- Λ at 98 per cent of the maximal growth rate as given by the present theory.

DISCUSSION

In the course of the analysis above it was necessary to make a number of assumptions. The most important one is the assumption of the applicability of the principle of minimum potential and surface tension energy. This principle is valid for stagnant, inviscid, incompressible fluids which are placed in a rigid container. These conditions are certainly well approximated in the experiments under consideration (for normal gravity) as shown above. Furthermore it is necessary that the system be completely isolated from its environment. This certainly is not the case in the heat transfer experiments: heat is transported through the fluids and gives rise to a phase change. The momentum transport at the gas–liquid interface which accompanies the evaporation can be neglected (according to the work of Moureau (1972)) since the experiments are performed under conditions which are far from the critical conditions. The energy contribution to the system due to heat transfer is transported through the system by means of the vapor bubbles rising upward. In short, for the experiments under consideration, the heat flux provides a constant flux of vapor necessary to reestablish the vapor film after bubbles detach from it. Essentially the same approach was taken by Lienhard & Wong (1964). A detailed justification for this approach can be found in the work of Hsieh (1972) on the effects of heat and mass transfer upon interfacial stability.

In the present analysis it is found that only the F_α waves explain the data well. They are characterized by a zero amplitude at the bottom of the cylinder (condition (50)). The waves actually observed in the experiments also satisfy this condition. The physical explanation for this may be that due to buoyancy and vapor generation in the vapor film, the cylindrical interface is somewhat displaced upwards from the axis of the heater. This causes the spacing between heater and film to be much smaller on the bottom of the heater and thus to make displacements there impossible.

In the analysis the thickness δ of the vapor film has been neglected. If one considers the actual radius of the interface $R + \delta$ instead of the radius of the heater, as one should, and if the Bond number is defined w.r.t. $R + \delta$ then the data as represented on figure 4 should all be shifted to the right over an amount proportional to δ/R. Expressions for δ as a function of the properties of the liquid and of the temperature difference between heater and liquid have been reported and evaluated by Baumeister & Hamill (1967). They find that e.g. for methanol $1 + \delta/R$ varies from 1.6 for $B = 0.01$ to 1.2 at $B = 0.5$ and less for larger values of B. For small values of B this results in a considerable shift of the data to the right, yet this results into somewhat better agreement with the present theory (see figure 4). For $B > 0.5$ the shift is much smaller and the good agreement between theory and experiment is retained. In addition it should be kept in mind that the predicted limits for Λ, based upon a 3% variation in Λ, will encompass all the data. In any case it should be observed that the experimental data straddles the predicted values of Λ, based upon the maximum value of ω_α even when the correction for the film thickness is taken into account.

CONCLUSIONS

A new variational approach to interfacial stability problems developed for stagnant fluids is applicable to fluids under the influence of a normal gravitational field. This approach leads to results which are in excellent agreement with experiment. The present work takes the three-dimensionality of the problem into account.

This variational stability analysis cannot be applied any more to a fluid in a centrifugal force field. A perturbation analysis indicates that Coriolis forces might lead to corrections.

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Résumé—On étudie la stabilité du film de vapeur formé, autour de cylindres horizontaux, au cours de l'ébullition par film stable. On compare la longueur d'onde de la perturbation possédant la plus forte vitesse de croissance à l'espacement observé pour les bulles se détachant de l'interface vapeur–liquide. Une approche variationnelle tridimensionnnelle conduit à des résültats qui sont en bon accord avec l'expérience pour un fluide dans un champ gravitationnel normal. On indique l'effet des forces de Coriolis sur les résultats obtenus dans un champ de forces centrifuge.

Auszug—Die Stabilitaet des Dampffilms auf wagerechten Zylindern beim stabilen Filmsieden wird untersucht. Die Wellenlaenge der am staerksten anwachsenden Stoerung wird mit dem beobachteten Abstand zwischen sich von der Dampf-Fluessigkeits-Trennschicht loesenden Blasen verglichen. Eine dreidimensionale Variationsmethode zeitigt Resultate, die mit Versuchen mit einer Fluessigkeit in einem normalen Schwerefeld gut uebereinstimmen. Der Einfluss von Coriolis-Kraeften auf in einem Feld von Zentrifugalkraeften erhaltenen Daten wird aufgezeigt.

Резюме—Исследована стабильность пленки паров, формирующейся в процессе устойчивого пленочного кипения на горизонтальных цилиндрах. Длина волны наиболее быстрого возмущения сравнена с наблюденным объемом пузырей, обособляющихся от границы раздела пар-жидкость. Трехмерный вариационный подход приводит к хорошо согласующимся с экспериментом относительно жидкости в обычном гравитационном поле результатам. Показано влияние кориоллисовых сил на данные, полученные в поле центробежных сил.